Lower Bounds for the Probability of a Union via Chordal Graphs

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We establish new Bonferroni-type lower bounds for the probability of a union of finitely many events where the selection of intersections in the estimates is determined by the clique complex of a chordal graph.

1 Introduction

The classical Bonferroni inequalities state that for any finite collection of events $\{A_v\}_{v\in V}$,

$$\begin{aligned} \Pr\left(\bigcup_{v\in V} A_v\right) &\le \sum_{\substack{I\in\mathscr{P}^*(V)\\ |I|\le 2r-1}} (-1)^{|I|-1}\Pr\left(\bigcap_{i\in I} A_i\right) \\ \Pr\left(\bigcup_{v\in V} A_v\right) &\ge \sum_{\substack{I\in\mathscr{P}^*(V)\\ |I|\le 2r}} (-1)^{|I|-1}\Pr\left(\bigcap_{i\in I} A_i\right) \end{aligned} \qquad (r=1,2,3,\ldots), \tag{1}$$

where $\mathscr{P}^*(V)$ denotes the set of non-empty subsets of V. Numerous variants of these inequalities are known, see, e.g., Galambos and Simonelli [6] for a survey.

In this paper, we establish a new variant where the selection of intersections in the probability bounds is determined by a chordal graph. Recall that a graph is *chordal* if it contains no cycles of length four or any higher length as induced subgraphs.

We refer to [2] for terminology on graphs. We write $G=(V,E)$ to denote that G is a graph having vertex-set V, which we assume to be finite, and edge-set E, consisting of two-element subsets of V. We use $\mathscr{C}(G)$ to denote the *clique complex* of G, that is, the abstract simplicial complex consisting of all non-empty cliques of G.

A recent connection between Bonferroni inequalities and chordal graphs is the chordal graph sieve, which states that in the classical Bonferroni upper bound, $\mathscr{P}^*(V)$ may be replaced by the clique complex of some arbitrary chordal graph on V. We recall the chordal graph sieve in the following proposition.

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Proposition 1.1 ([3, 4]). *Let $\{A_v\}_{v\in V}$ be a finite collection of events, where the indices form the vertices of a chordal graph G. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \le \sum_{I\in\mathscr{C}(G)} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right), \tag{2}$$

and

$$\Pr\left(\bigcup_{v\in V} A_v\right) \le \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\le 2r-1}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \quad (r = 1,2,3,\dots). \tag{3}$$

Remark 1.2. Inequality (2) interpolates between Boole's inequality (G edgeless) and the sieve formula (G complete). Moreover, it generalizes a well-known result of Hunter [9] on trees. Note that for any tree $G = (V,E)$, the clique complex of G consists of the vertices and edges of G.

Remark 1.3. The case $r = 2$ in (3) was independently found by Boros and Veneziani [1] by linear programming techniques; see also [16] for a discussion on this bound.

2 Main result

In the sequel, we use $c(G)$ resp. $\alpha(G)$ to denote the number of connected components of G resp. the independence number of G. Our main result, which is a lower bound analogue of Proposition 1.1, reads as follows.

Theorem 2.1. *Let $\{A_v\}_{v\in V}$ be a non-empty finite collection of events, where the indices form the vertices of a chordal graph G. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \ge \frac{1}{\alpha(G)} \sum_{I\in\mathscr{C}(G)} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right), \tag{4}$$

and

$$\Pr\left(\bigcup_{v\in V} A_v\right) \ge \frac{1}{\alpha(G)} \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\le 2r}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \quad (r = 1,2,3,\dots). \tag{5}$$

Remark 2.2. As an immediate consequence of (2), and in contrast to the classical Bonferroni inequalities, the lower bound in (4) is guaranteed to be non-negative.

Remark 2.3. Inequality (4) becomes an equality if G is complete. In this case, (4) agrees with the sieve formula, while (5) coincides with the usual Bonferroni lower bounds.

The proof of Theorem 2.1 is facilitated by a lemma and a proposition. The lemma contains a well-known combinatorial identity.

Lemma 2.4 ([13]). *For any $n \in \mathbb{N}$ and any $m \in \mathbb{N}_0$,*

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}. \tag{6}$$

Proposition 2.5. *For any chordal graph $G = (V, E)$,*

$$\sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} \leq c(G) \quad (r = 1, 2, 3, \dots), \tag{7}$$

with equality if $r \geq |V|/2$.

Proof. It suffices to show that (7) holds for any connected chordal graph G, that is,

$$\sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} \leq 1 \quad (r = 1, 2, 3, \dots), \tag{8}$$

with equality if $r \geq |V|/2$. If G has at most one vertex, then this statement trivially holds. We proceed by induction on the number of vertices of G. If G has at least two vertices, then since G is chordal, G contains a simplicial vertex s, that is, a vertex whose neighborhood $N_G(s)$ is a clique. Since G is connected, $N_G(s) \neq \emptyset$. Moreover, since s is simplicial, the subgraph $G - s$ obtained from G by removing s is connected. Since, in addition, $G - s$ is chordal, we may apply the induction hypothesis to $G - s$, whence

$$\begin{aligned} \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} &= \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r \\ s \notin I}} (-1)^{|I|-1} + \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r \\ s \in I}} (-1)^{|I|-1} = \sum_{\substack{I \in \mathscr{C}(G-s) \\ |I| \leq 2r}} (-1)^{|I|-1} + \sum_{\substack{I \subseteq N_G(s) \\ |I| \leq 2r-1}} (-1)^{|I|} \\ &\leq 1 + \sum_{k=0}^{2r-1} (-1)^k \binom{|N_G(s)|}{k} = 1 + (-1)^{2r-1} \binom{|N_G(s)| - 1}{2r-1} \leq 1, \end{aligned} \tag{9}$$

where the equality in (9) follows from (6). Note that if $r \geq |V|/2$, then by the induction hypothesis, the first inequality in (9) becomes an identity. The same applies to the second inequality, since for $r \geq |V|/2$ the binomial coefficient $\binom{|N_G(s)|-1}{2r-1}$ vanishes. □

Remark 2.6. Eq. (8) can also be proved by combining the following topological results:

1. The clique complex of any connected chordal graph is contractible ([5]).
2. For any contractible abstract simplicial complex $\mathscr{S}$, the Euler characteristic of its $(2r-1)$-skeleton is at most 1, with equality if $r \geq |V|/2$ ([14]).

We are now ready to proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1. It suffices to prove (5), since (4) follows from (5) by choosing some arbitrary $r \geq |V|/2$. For any $I \subseteq V$, $I \neq \emptyset$, define

$$B_I := \bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} \overline{A_i}. \tag{10}$$

Note that the B_I's form a partition of $\bigcup_{v \in V} A_v$, that is,

$$\bigcup_{v \in V} A_v = \bigcup_{\emptyset \neq I \subseteq V} B_I \text{ and } B_I \cap B_J = \emptyset \quad (I \neq J).$$

Clearly, $G[J]$ is chordal and $c(G[J]) \leq \alpha(G)$ for any $J \subseteq V$, $J \neq \emptyset$. Hence, by applying Proposition 2.5 to $G[J]$ we obtain

$$\begin{aligned}
\Pr\left(\bigcup_{v \in V} A_v\right) &= \Pr\left(\bigcup_{\substack{J \subseteq V \\ J \neq \emptyset}} B_J\right) = \sum_{\substack{J \subseteq V \\ J \neq \emptyset}} \Pr(B_J) \geq \frac{1}{\alpha(G)} \sum_{\substack{J \subseteq V \\ J \neq \emptyset}} c(G[J]) \Pr(B_J) \\
&\geq \frac{1}{\alpha(G)} \sum_{\substack{J \subseteq V \\ J \neq \emptyset}} \sum_{\substack{I \in \mathscr{C}(G[J]) \\ |I| \leq 2r}} (-1)^{|I|-1} \Pr(B_J) = \frac{1}{\alpha(G)} \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} \sum_{J \supseteq I} \Pr(B_J) \\
&= \frac{1}{\alpha(G)} \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcup_{J \supseteq I} B_J\right) = \frac{1}{\alpha(G)} \sum_{\substack{I \in \mathscr{C}(G) \\ |I| \leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcap_{i \in I} A_i\right). \qquad \square
\end{aligned}$$

Remark 2.7. In view of the preceding proof, the bounds can be sharpened by replacing $\alpha(G)$ with $\alpha'(G) := \max\{c(G[J]) \mid J \subseteq V, J \neq \emptyset, B_J \neq \emptyset\}$ where B_I is defined as in (10). In particular, if the underlying graph matches the structure of the events in such a way that $G[J]$ is connected whenever B_J is non-empty, then $\alpha'(G) = 1$ and the bounds in (2) and (4) coincide. The resulting identity is well-known in abstract tube theory [3].

Remark 2.8. The requirement that G is chordal cannot be omitted from Theorem 2.1. Consider the non-chordal graph $G = (V, E)$ depicted in Figure 1, and consider events A_v, $v \in V$, with $\Pr(A_v) = 1$ for any $v \in V$. The clique complex of this graph consists of 8 cliques of size 1, 20 cliques of size 2, and 16 cliques of size 3. Since $\alpha(G) = 3$, the first inequality in Theorem 2.1 gives the non-valid bound $1 \geq \frac{1}{3}(8 - 20 + 16) = \frac{4}{3}$.

Remark 2.9. The graph G in the preceding counterexample was obtained as a subgraph of G_3 — the first graph in an infinite sequence $(G_k)_{k=3,5,\ldots}$ of non-chordal graphs for which the hypothesis of Theorem 2.1 does not hold. Each G_k in this sequence is the join of k disjoint copies of the edgeless graph on three vertices. It turns out that G_k has independence number 3, and that the clique complex of G_k has Euler characteristic $1 + 2^k$. By considering events A_v, $v \in V$, with $\Pr(A_v) = 1$ for any vertex v of G_k, the first inequality in Theorem 2.1 becomes $1 \geq \frac{1}{3}(1 + 2^k)$, which is false for $k = 3, 5, \ldots$.

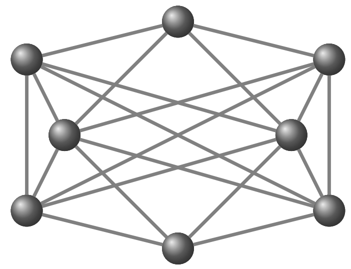

Figure 1: A non-chordal graph for which the hypothesis of Theorem 2.1 does not hold.

Theorem 2.10. *Under the requirements of Theorem 2.1,*

$$\sum_{I\in\mathscr{C}(G)} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \geq \sum_{\substack{I\in\mathscr{C}(G)\\|I|\leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \quad (r = 1,2,3,\dots).$$

Proof. In view of the proof of Theorem 2.1, it suffices to show that for any $J \subseteq V$, $J \neq \emptyset$,

$$\sum_{I\in\mathscr{C}(G[J])} (-1)^{|I|-1} \geq \sum_{\substack{I\in\mathscr{C}(G[J])\\|I|\leq 2r}} (-1)^{|I|-1}.$$

A two-fold application of Proposition 2.5 reveals that the sum on the left-hand side equals $c(G[J])$, while the sum on the right-hand side is bounded above by $c(G[J])$. $\square$

3 Particular cases

In case of a tree, Theorem 2.1 gives a lower bound analogue of Hunter's inequality [9]. Hunter's upper bound on $\Pr(\bigcup_{v\in V} A_v)$ coincides with the bracketed term in (11).

Corollary 3.1. *Let $\{A_v\}_{v\in V}$ be a non-empty finite collection of events, where the indices form the vertices of a tree $G = (V,E)$. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{\alpha(G)}\left(\sum_{v\in V}\Pr(A_v) - \sum_{\{v,w\}\in E}\Pr(A_v\cap A_w)\right). \tag{11}$$

Remark 3.2. Since $\alpha(G) \leq n-1$ for any tree G on $n > 1$ vertices, we conclude from (11) that, in this case,

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{n-1}\left(\sum_{v\in V}\Pr(A_v) - \sum_{\{v,w\}\in E}\Pr(A_v\cap A_w)\right).$$

To obtain a maximum lower bound of this type, we adapt Hunter's method [9] for computing a minimum upper bound: Any minimum spanning tree of the complete

weighted graph on V with edge weights $\Pr(A_v \cap A_w)$ for any $v, w \in V$, $v \neq w$, maximizes the right-hand side of the preceding inequality. Such a minimum spanning tree can be found in polynomial time, e.g., by applying Kruskal's algorithm [10].

Remark 3.3. If G is a path on n vertices, then $\alpha(G) = \lceil n/2 \rceil$ and hence,

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{\lceil n/2 \rceil}\left(\sum_{v\in V} \Pr(A_v) - \sum_{\{v,w\}\in E} \Pr(A_v \cap A_w)\right). \tag{12}$$

In order to find a path on V that maximizes the lower bound in (12), consider the complete graph on V with edge lengths $\Pr(A_v \cap A_w)$ for any $v, w \in V$, $v \neq w$. Any minimum length Hamiltonian path in this graph maximizes the right-hand side of (12). Unfortunately, the problem of finding a minimum length Hamiltonian path is NP-hard [8].

Corollary 3.4. *Let $A_1, \ldots, A_n$ be a non-empty finite collection of events. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{\lceil n/2 \rceil}\left(\sum_{i=1}^{n} \Pr(A_i) - \frac{2}{n}\sum_{1\leq i<j\leq n} \Pr(A_i \cap A_j)\right). \tag{13}$$

Proof. The inequality follows from (12) by averaging over all paths on $V = \{1, \ldots, n\}$. □

Remark 3.5. Inequality (13) may be considered as a lower bound analogue of Kwerel's inequality [11], which states that $\Pr\left(\bigcup_{v\in V} A_v\right) \leq \sum_{i=1}^{n} \Pr(A_i) - \frac{2}{n}\sum_{1\leq i<j\leq n} \Pr(A_i \cap A_j)$.

Our next corollary provides a lower bound analogue of Seneta's inequality [15]. The upper bound in Seneta's inequality coincides with the bracketed term that follows.

Corollary 3.6. *Let $A_1, \ldots, A_n$ be a finite collection of events, and $j, k \in \{1, \ldots, n\}$. Then,*

$$\begin{aligned}\Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \frac{1}{n-2+\delta_{jk}}\Bigg(&\sum_{i=1}^{n} \Pr(A_i) - \sum_{\substack{i=1\\ i\neq j}}^{n} \Pr(A_i \cap A_j) - \sum_{\substack{i=1\\ i\neq j,k}}^{n} \Pr(A_i \cap A_k)\\ &+ \sum_{\substack{i=1\\ i\neq j,k}}^{n} \Pr(A_i \cap A_j \cap A_k)\Bigg),\end{aligned}$$

provided $n > 2 - \delta_{jk}$ where δ_{jk} denotes the Kronecker delta.

Proof. Define $G = K_{2-\delta_{jk}} * L_{n-2+\delta_{jk}}$ (the join of a complete graph on $2 - \delta_{jk}$ vertices and an edgeless graph on $n - 2 + \delta_{jk}$ vertices), and apply Theorem 2.1. □

By taking the average over all distinct $j, k = 1, \ldots, n$ in the preceding inequality, a lower bound analogue of another result due to Kwerel [12] is obtained:

Corollary 3.7. *Let $A_1,\dots,A_n$ be a finite collection of events where $n \geq 3$. Then,*

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \frac{1}{n-2}\left(\sum_{i=1}^{n}\Pr(A_i) - \frac{2n-3}{\binom{n}{2}}\sum_{\substack{i,j=1\\ i<j}}^{n}\Pr(A_i\cap A_j) + \frac{3}{\binom{n}{2}}\sum_{\substack{i,j,k=1\\ i<j<k}}^{n}\Pr(A_i\cap A_j\cap A_k)\right).$$

The following corollary generalizes Corollary 3.7. Its upper bound analogue [3, 4] is related to an inequality of Galambos and Xu [7].

Corollary 3.8. *Let $A_1,\dots,A_n$ be events. Then, for $m = 0,\dots,n-1$ we have*

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \frac{1}{n-m}\left(\sum_{k=1}^{m}(-1)^{k-1}\frac{\binom{m}{k}}{\binom{n}{k}}\cdot\frac{nk-(m+1)(k-1)}{m-k+1}\sum_{i_1<\dots<i_k}\Pr\left(A_{i_1}\cap\dots\cap A_{i_k}\right)\right.$$
$$\left. + (-1)^m\frac{m+1}{\binom{n}{m}}\sum_{i_1<\dots<i_{m+1}}\Pr\left(A_{i_1}\cap\dots\cap A_{i_{m+1}}\right)\right). \quad (14)$$

Proof. For any $M \subseteq \{1,\dots,n\}$ let G_M denote the join of the complete graph on M and the edgeless graph on the complement of M. It follows by induction on $|M|$ that each G_M is chordal. By averaging the right-hand side of (4) over all G_M with $|M| = m$ we obtain

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \frac{1}{n-m}\left(\frac{1}{\binom{n}{m}}\sum_{\substack{M\subseteq\{1,\dots,n\}\\ |M|=m}}\ \sum_{\substack{I\in\mathscr{P}^*(\{1,\dots n\})\\ I \text{ clique of } G_M}}(-1)^{|I|-1}\Pr\left(\bigcap_{i\in I}A_i\right)\right). \quad (15)$$

Now follow the lines of the proof of Theorem 4.4.7 in [3, pp. 42–43], where it is shown that the bracketed terms in (14) and (15) are equal. Thus, the corollary is proved. ☐

4 Open problems

Normally, when applying Bonferroni inequalities, joint probabilities of up to k events are given, and reasonable estimates on the probability of their union are wanted. To obtain a minimum upper bound, respectively maximum lower bound on this probability based on the inequalities in (2–5), a chordal graph having clique number k or less is sought which minimizes, respectively maximizes the bound on the right-hand side of these inequalities. This seems to be an intrinsic computational problem. Even in the particular case of a tree, where Hunter's method [9] gives a polynomial time algorithm for obtaining a minimum upper bound, no efficient algorithm is known for maximizing the bound in (11). We leave these computational problems for future research.

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Appendix: Application to Network Reliability

Consider the directed network N in Figure 2(a), in which all nodes are perfectly reliable, and the arcs are subject to random and independent failure where all failure probabilities are assumed to be known. We are interested in lower bounds on its *source-to-terminal reliability* $R_{st}(N)$, which is the probability that there is a directed path from s and t consisting of non-failing arcs only. Obviously, this is the case if all arcs in the directed paths 15, 136, 245, 26 are operating. For this ordering of s,t-paths, let A_i denote the event that all arcs in the ith path are operating ($i = 1,\ldots,4$). Then, we have

$$R_{st}(N) = \Pr(A_1 \cup A_2 \cup A_3 \cup A_4).$$

If we assume that the reliability of each arc is p, then by the sieve formula,

$$R_{st}(N) = 2p^2 + 2p^3 - 5p^4 + 2p^5. \tag{16}$$

Corollary 3.1 (for $G = 1 - 2 - 3 - 4$) and Corollary 3.4 give

$$R_{st}(N) \geq p^2 + p^3 - p^4 - \tfrac{1}{2}p^6, \tag{17}$$

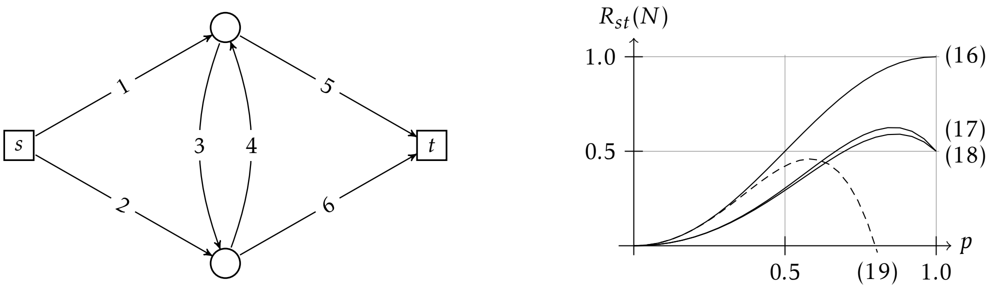
$$R_{st}(N) \geq p^2 + p^3 - \tfrac{5}{4}p^4 - \tfrac{1}{4}p^6. \tag{18}$$

In order to compare these bounds with the classical Bonferroni bound of degree 2, we put $r = 1$ in (1). In this way, we obtain

$$R_{st}(N) \geq 2p^2 + 2p^3 - 5p^4 - p^6. \tag{19}$$

Figure 2(b) shows the reliability function (16) and the lower bounds (17–19) for any p in the unit interval $[0,1]$. It turns out that for large values of p, the bounds in (17) and (18) are closer to the exact reliability value than the classical Bonferroni bound.

From a practical point of view, larger values of p are more relevant than lower values of p, since network components are usually highly reliable.



(a) Network N with terminal nodes s and t.

(b) Lower bounds for $R_{st}(N)$.

Figure 2: Application to network reliability analysis.